4 [2.05, 3, 5].—LOTHAR COLLATZ & WOLFGANG WETTERLING, *Optimierungsaufgaben*, Springer-Verlag, Berlin, 1966, ix + 181 pp., 21 cm. Price DM 10.80.

This book provides a clear and readable introduction into the fundamental principles of linear and convex programming, as well as the theory of matrix games. These principles provide a framework for a theory of Chebyshev approximations with applications to elliptic differential equations. This part of the book should be of particular interest. The difficult problems connected with the minimization of convex functions without constraints are not touched.

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5 [2.05].—MIECZYSLAW WARMUS, *Tables of Lagrange Coefficients for Quadratic Interpolations*, Polish Scientific Publishers, Warsaw, 1966, ix + 501 pp., 30 cm. Price Zl 180.

This volume, the second in a series of mathematical tables prepared at the Computing Centre of the Polish Academy of Sciences, gives values of the Lagrange interpolation coefficients $L_{-1}(t) = -t(1 - t)/2$, $L_1(t) = t(1 + t)/2$ to 11D; and $L_0(t) = 1 - t^2$ to 10D, all for $t = 0(0.00001)1$.

These tables are arranged in a condensed form, using the relations $L_{-1}(1 - t) = L_1(t)$, $L_0(1 - t) = L_0(t)$, and $L_1(1 - t) = L_{-1}(t)$.

Herein the argument-interval is one-tenth that of the previously largest similar table [1] and two more decimal places appear in each of the tabular entries.

The author points out in the preface that these tables provide an easy method of calculating the value of a function corresponding to an argument given to $k + 5$ decimal places from tabular values for arguments given to k decimal places, and he illustrates this with a single numerical example, which includes an estimate of the error arising from such interpolation.

The procedure followed in the calculation of these tables is not discussed, and no bibliography of earlier tables is given.

It seems appropriate to this reviewer to mention here the equally voluminous, unpublished 8D tables of Salzer & Richards [2] for quadratic and cubic interpolation by the Gregory-Newton and Everett formulas.

J. W. W.

1. NYMTP, *Tables of Lagrangian Interpolation Coefficients*, Columbia Univ., New York, 1944. (See *MTAC*, v. 1, 1943–1945, pp. 314–315, RMT **162**.)
2. HERBERT E. SALZER & CHARLES H. RICHARDS, *Tables for Non-linear Interpolation*, 1961. Copy deposited in UMT file. (See *Math. Comp.*, v. 16, 1962, p. 379, RMT **31**.)

6 [2.10, 3, 6, 7].—R. E. BELLMAN, R. E. KALABA & J. LOCKETT, *Numerical Inversion of the Laplace Transform*, American Elsevier Publishing Co., Inc., New York, 1966, viii + 249 pp., 24 cm. Price $9.50.

In numerous applied problems, characterized by ordinary differential equations, difference-differential equations, partial differential equations or other functional equations, the Laplace transform is often a powerful tool for obtaining a solution.

When the Laplace transform approach is applicable, getting the Laplace transform of the solution is relatively easy. The major problem is inverting the transform. It is often the case that closed-form representations in terms of tabulated functions for the inverse are not known, and so one must resort to numerical methods.

A comprehensive volume on the numerical inversion of Laplace transforms replete with examples would fill an important gap in the literature. Although the volume under review has much which is commendable, it is not comprehensive in its coverage, as the authors seem completely unaware of important segments of the literature. We return to this point later, but first we explore the contents of the volume and present a generalization of the basic tool used by the authors.

Consider

$$h(p) = \int_0^\infty e^{-pt} f(t)dt \tag{1}$$

which we assume exists. Given $h(p)$ the Laplace transform of $f(t)$, the problem is to find $f(t)$. Elementary properties of $h(p)$ are treated in Chapter 1 and its numerical inversion is considered in Chapter 2. The volume is centered around a procedure which treats (1) as an integral equation. This is a useful approach since the techniques are applicable to the solution of linear integral equations of the form

$$u(x) = f(x) + \int_a^b k(x, y)u(y)dy\,. \tag{2}$$

The idea is to approximate the integral in (1) by a quadrature formula and then find approximate values of $f(t)$ by solving a system of linear equations. By appropriate choice of a quadrature formula, the solution can be expressed in a neat form without actually having to solve the system of linear equations in the usual sense. In (1), put

$$x = e^{-vt}, \qquad v > 0\,, \qquad vt = -\ln x\,, \tag{3}$$

so that

$$vh(p) = \int_0^1 x^{(p/v)-1} f(t)dx\,. \tag{4}$$

From the theory of orthogonal functions [1], we have

$$h(p) = \sum_{j=1}^{n} w_j x_j^{(p/v)-1} z(x_j) + F_n\,, \tag{5}$$

where

$$\begin{gathered} w_j = \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\Gamma(n+\lambda)x_j^{\beta+1}(1-x_j)^{\alpha+1}v[R_n^{(\alpha,\beta)\prime}(x_j)]^2}\,, \qquad \lambda = \alpha+\beta+1\,, \\ R_n^{(\alpha,\beta)}(x_j) = 0\,, \qquad x_j = e^{-vt_j}\,, \qquad z(x_j) = f(t_j)\,, \end{gathered} \tag{6}$$

where F_n is a remainder term and $R_n^{(\alpha,\beta)}(x)$ is the shifted Jacobi polynomial, i.e., $R_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(2x - 1)$. Let p take on n distinct values and put $F_n = 0$. Then (4) gives rise to n equations in n unknowns. By a judicious choice of the p values, the solution of the linear equation system with $F_n = 0$ is easily constructed. With

$$
\begin{aligned}
(7) \qquad & p/v = k + 1\,, \qquad k = 0, 1, \cdots, n - 1\,, \\
& a_k = h(v[k+1])\,, \qquad w_j z(x_j) = y_j
\end{aligned}
$$

the linear equation system to solve is

$$
(8) \qquad \sum_{j=1}^{n} x_j^k y_j = a_k\,.
$$

It can be shown that

$$
(9) \qquad
\begin{aligned}
w_j z(x_j) &= \sum_{k=0}^{n-1} a_k q_{k,j}\,, \\
\sum_{k=0}^{n-1} q_{k,j} x^k &= \frac{R_n^{(\alpha,\beta)}(x)}{(x - x_j) R_n^{(\alpha,\beta)\prime}(x_j)}\,.
\end{aligned}
$$

The authors treat the case $\alpha = \beta = 0$ only, in which event $P_n^{(0,0)}(x)$ is the Legendre polynomial. To facilitate use of the formulas, tables of x_j, $R_n^{(0,0)\prime}(x_j)$ and $q_{k,j}$ are provided for $n = 3(1)15$ to 17S, of which the first 15 figures are believed to be correct.

Observe that in the above general development, the parameters v, α and β are free, and ideally this should be exploited to smooth out irregularities in the behavior of the functions involved and so improve the efficiency of the inversion process. This points up a shortcoming in the analysis, since for each choice of α and β tables of x_j, $R_n^{(\alpha,\beta)\prime}(x_j)$ and $q_{k,j}$ must be prepared. Use of the Chebyshev polynomials $T_n^*(x)$ and $U_n^*(x)$ (except for normalization constants, these are the cases $\alpha = \beta = -\frac{1}{2}$ and $\alpha = \beta = \frac{1}{2}$, respectively) would shorten some of the computational effort, since the zeros x_j and weights w_j are easily expressed. Of course, other orthogonal polynomials with weights chosen to reflect singularities could also be used.

Several examples are treated in Chapters 3 and 4 to show that the procedure for $\alpha = \beta = 0$ can lead to satisfactory results. A number of examples are also developed to show how the ideas may be extended to solve other functional equations, both linear and nonlinear. It is a virtue of the volume that it warns the reader that there is no panacea and that pitfalls abound. The point is this. If (8) is expressed in matrix form as $Ay = b$, A^{-1}, of course, is known from (9). However, there may be serious difficulties as the matrix A is ill-conditioned. Thus $A^{-1}b$ may be meaningless unless b is known to high accuracy. Chapter 5 studies applications of dynamic programming techniques for the solution of ill-conditioned systems. It would be interesting to know if the detrimental effects of ill-conditioning can be removed or mitigated by use of other choices of α and β.

An appendix lists some FORTRAN IV programs for "The Heat Equations," "Routing Problem" and "Adaptive Computation." These pages are a total loss, as I do not find any information as to which specific problems the programs apply.

I now return to the statement made earlier concerning material which deserves a place in a comprehensive treatment on the inversion of transforms. We divide our discussion, which is necessarily brief and by no means complete, into three parts.

First, there are two important papers by A. Erdélyi [2], [3]. There it is shown that if $f(t)$ in (4) is expanded in a series of the Jacobi polynomials $R_n^{(\alpha,\beta)}(t)$, then the coefficients in this expansion, call them a_n, can be expressed as a finite sum of

$n + 1$ terms, each of which depends on a different value of $h(p)$. Notice that this procedure yields a continuous-type approximation as opposed to the discrete-type approximation described by [2]–[9]. Attention should also be called to papers by Tricomi [4], who got a continuous-type approximation based on Laguerre polynomials.

In the above approaches, the problem is viewed as that of solving an integral equation. As the inverse Laplace transform has an integral representation, it is natural to seek the inverse transform by a direct quadrature. Examples of this approach are given in three papers by Salzer [5], [6], [7].

Finally, we note a valuable technique which is slightly touched upon by the authors. However, no references to the literature are given. The idea is to approximate $h(p)$ by the ratio of two polynomials and then invert this approximation in the usual fashion. Only a few examples of this approach are known; see the papers by Luke [8]–[10] and a paper by Fair [11]. In each instance the accuracy of the results is quite remarkable. Furthermore, the approximation for $f(t)$ is a sum of exponentials. This is especially valuable in numerous problems where integrals and other expressions involving $f(t)$ are required.

Y. L. L.

1. A. Erdélyi, W. Magnus, F. Oberhettinger & F. G. Tricomi, *Higher Transcendental Functions*, Vol. 2, McGraw-Hill, New York, 1953.
2. A. Erdélyi, "Inversion formulae for the Laplace transformation," *Philos. Mag.*, (7), v. 34, 1943, pp. 533–536.
3. A. Erdélyi, "Note on an inversion formula for the Laplace transformation," *J. London Math. Soc.*, v. 18, 1943, pp. 72–77.
4. F. G. Tricomi, "Transforzione di Laplace e polinomi di Laguerre," *Rend. Accad. Naz. dei XL*, (6), v. 13, pp. 232–239, 420–426.
5. H. E. Salzer, "Equally-weighted quadrature formulas for inversion integrals," *MTAC*, v. 11, 1957, pp. 197–200.
6. H. E. Salzer, "Tables for the numerical calculation of inverse Laplace transforms," *J. Math. and Phys.*, v. 37, 1958, pp. 89–109.
7. H. E. Salzer, "Additional formulas and tables for orthogonal polynomials originating from inversion integrals," *J. Math. and Phys.*, v. 40, 1961, pp. 72–86.
8. Y. L. Luke, "Rational approximations to the exponential function," *J. Assoc. Comput. Mach.*, v. 4, 1957, pp. 24–29.
9. Y. L. Luke, "On the approximate inversion of some Laplace transforms," *Fourth U. S. Congr. Appl. Mech.*, 1962, Amer. Soc. Mech. Engrs., New York, pp. 269–276.
10. Y. L. Luke, "Approximate inversion of a class of Laplace transforms applicable to supersonic flow problems," *Quart. J. Mech. Appl. Math.*, v. 17, 1964, pp. 91–103.
11. W. Fair, "Padé approximation to the solution of the Riccati equation," *Math. Comp.*, v. 18, 1964, pp. 627–634.

7 [2.35, 4, 5, 6, 13.15].—Yu. V. Vorobyev, *Method of Moments in Applied Mathematics*, translated from Russian by B. Seckler, Gordon and Breach Science Publishers, New York, 1965, x + 165 pp., 23 cm. Price $12.50.

This monograph presents a study with applications of the method of moments for the approximate solution of functional equations in Hilbert spaces involving (mostly completely continuous and self-adjoint bounded) linear operators. The method is based on a variational principle and is closely related to the Chebyshev-Markov classical problem of moments. The representation of the approximate operators constructed in the method of moments shows that the author's method falls within the general framework of the projection or the abstract Ritz-Galerkin method. It differs merely in the choice of the projections, that is, the method of moments gives a specific and very often a useful way of determining the coordinate elements